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Notes on Semi-Implicit Modeling Techniques

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Notes on Semi-Implicit Modeling Techniques

One of the most severe constraints in semi-implicit modeling is related to the requirement of solution of a system of Helmholtz equations. The usual method of solution by relaxation methods requires large horizontal fields of data in the computer core storage at one time, while other parts of the computation are more conveniently performed on vertical strips of data. This dual form of data use leads to inefficiencies in data access and greater wall clock times for forecast programs.

As suggested in notes by D. Burridge of the European Meteorological Center, a program has been written to test a code organization that allows all calculations in strip form. Basic to this method are (1) that the system of Helmholtz equations is separated in the vertical to allow their independent solution, and (2) they are solved by tri-diagonal block eliminations. The elimination is ordered so that the calculations are performed alternately in the north-to-south and south-to-north directions.

In order to test these ideas, a model with only the vertical and one horizontal dimension has been designed. It is used in this note to illustrate the solution method suggested by D. Burridge, and the advantages and disadvantages of such a method. Pressure is the vertical coordinate. The earth's surface is assumed flat. In the usual fashion, the pressure gradient, divergence, and vertical velocity are averaged in time. The finite-difference equations are:

$$\overline{u}^{2t}_k = u_k^{\tau-1} - \Delta t\left(\overline{\overline{v}^{y} u_y}^{y} + \overline{\overline{\omega}^{y} u_p}^{p} - f\, \overline{v}^{yy} \right)^{\tau} = U_k^{*} \qquad (1)$$

$$\overline{v}^{2t}_k + \tfrac{1}{2}\Delta t\left(\overline{\phi}^{2t}_{ky} + \overline{\phi}^{2t}_{k+1_y} \right) = v_k^{\tau-1} - \Delta t\left(\overline{\overline{v}^{y} v_y}^{y} + \overline{\overline{\omega}^{y} v_p}^{p} + f\, \overline{u}^{yy} \right) = V_k^{*} \qquad (2)$$

$$\overline{\theta'}^{2t}_k + \tfrac{1}{2}\Delta t\, \overline{\theta}_{k_p}\left(\overline{\omega}^{2t}_k + \overline{\omega}^{2t}_{k+1} \right) = \theta'^{\tau-1}_k - \Delta t\left(\overline{\overline{v}^{yy} \theta'_y}^{y} + \overline{\overline{\omega}^{y}\ \overline{\theta'}^{y}_p}^{yp} \right) = T_k^{*} \qquad (3)$$

$$\overline{\phi}^{2t}_{k+1} - \overline{\phi}^{2t}_k + c_p \Delta\pi_k \overline{\theta'}^{2t}_k = -\beta\, c_p\, \overline{\theta}_k \Delta\pi_k = \beta\, G_k \qquad (4)$$

$$\overline{\phi}^{2t}_1 - \Delta t\, \alpha_s \overline{\omega}^{2t}_1 = \phi_1^{\tau-1} - \Delta t\, \overline{\overline{v}^{yy} \phi_{1_y}}^{y} = G_1^{*} \qquad (5)$$

$$\Delta p_k\, \overline{v}^{2t}_{k_y} + \overline{\omega}^{2t}_{k+1} - \overline{\omega}^{2t}_k = 0 \qquad (6)$$

where usual finite-difference notation is used. In addition

$$\overline{(\ \ )}^{2t} = \tfrac{1}{2}[(\ \ )^{\tau-1} + (\ \ )^{\tau-1}] \ . \qquad (7)$$

The superscript, τ, refers to the time level. Also,

$$\overline{\theta}_{k_p} = \frac{d\, \overline{\theta}_k}{dp} \qquad (8)$$

is the derivative of the basic state potential temperature with respect to p in the kth layer. Also

$$\theta_k = \overline{\theta}_k + \theta'_k \ . \qquad (9)$$

If $\beta = 0$, then all ϕ_k's are replaced by ϕ'_k, where

$$\phi_k = \phi_{o_k} + \phi'_k \ . \qquad (10)$$

If $\beta = 1$, then ϕ_k's are the total value. The reason for this distinction will be seen later. It is also noted that π is the Exner function,

$$\pi = \left(\frac{p}{p_o}\right)^{\kappa} \qquad (11)$$

and α_s is the surface specific volume. The data placement is shown in Figure 1.

Equations (1)-(6) cannot be marched directly ahead in time. Instead, fields must mutually satisfy all the equations. The solution is obtained by eliminating all variables, except $\overline{\phi}_k^{2t}$ between the equations. It is convenient to introduce the notation

$$\overline{\phi}_k = \overline{\phi}_k^{2t} + \overline{\phi}_{k+1}^{2t} . \tag{12}$$

The resulting equation in this variable is

$$\overline{\phi}_k + 2(-1)^k \sum_{j=1}^{k-1} (-1)^j \, \overline{\phi}_j + (-1)^k \sum_{j=1}^{L} a_j \, \overline{\phi}_{jyy}$$

$$+ b_k\left(\Delta p_k \overline{\phi}_{kyy} + 2 \sum_{j=k+1}^{L} \Delta p_j \overline{\phi}_{jyy}\right) = X_k, \; k = 1,2,\ldots,L \tag{13}$$

where the number of layers is L, and

$$a_k = - (\Delta t)^2 \, \alpha_s \Delta p_k \tag{14}$$

$$b_k = \tfrac{1}{4}(\Delta t)^2 \, \Delta\pi_k \theta_{kp} c_p \tag{15}$$

and

$$X_k = \beta \, G_k - 2(-1)^k \, [G^*_1 + \Delta t \, \alpha_s \sum_{j=1}^{L} \Delta p_j V^*_{jy}]$$

$$- c_p \Delta\pi_k \{T^*_k - \tfrac{1}{2} \, \Delta t \, \overline{\theta}_{kp} [\Delta p_k V^*_{ky} + 2 \sum_{j=k+1}^{L} \Delta p_j V^*_{jy}]\} \tag{16}$$

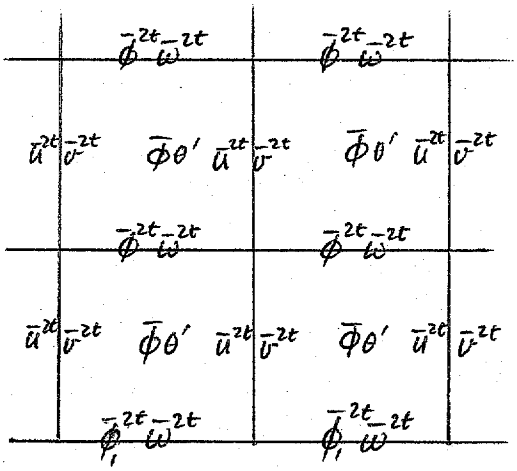
$\bar{\phi}^{2t}$ $\bar{\omega}^{2t}$
$\bar{u}^{2t}$ $\bar{v}^{2t}$
$\bar{\Phi}\theta'$
$\bar{\phi}_1^{2t}$ $\bar{\omega}^{2t}$

Fig. 1. Data placement

Once $\overline{\phi}_k$ is determined from (13) (more about that in awhile) then the other variables can be determined from

$$\overline{u}_k^{2t} = U_k^* \tag{17}$$

$$\overline{v}_k^{2t} = V_k^* - \tfrac{1}{2}\,\Delta t\,\overline{\phi}_{ky} \tag{18}$$

$$\overline{\omega}_k^{2t} = \overline{\omega}_{k+1}^{2t} + p_k\overline{v}_{ky}^{2t} \tag{19}$$

with

$$\overline{\omega}_{L+1}^{2t} = 0 \tag{20}$$

$$\overline{\phi}_1^{2t} = \Delta t\,\alpha_s\,\overline{\omega}_1^{2t} + G_1^* \tag{21}$$

$$\overline{\theta'}_k^{2t} = T_k^* - \tfrac{1}{2}\,\Delta t\,\overline{\theta}_{k_p}(\overline{\omega}_k^{2t} + \overline{\omega}_{k+1}^{2t}) \tag{22}$$

$$\overline{\phi}_{k+1}^{2t} = \overline{\phi}_k - \overline{\phi}_k^{2t} + (1-\beta)\left(\phi_{o_k} + \phi_{o_{k+1}}\right) \tag{23}$$

The values of quantities at $\tau+1$ are obtained from

$$(\ )^{\tau+1} = 2\overline{(\ )}^{2t} - (\ )^{\tau-1} \tag{24}$$

Solution to the Vertical Problem

The method of solution to (13) will be split into two parts. As it stands, this equation is a Helmholtz equation involving both the horizontal and vertical coordinates. In the general three-dimensional problem, two horizontal directions would be involved. It is desirable to separate the problem into a series of Helmholtz problems, one for each layer. Sela and Scolnik (1972) describe a way of doing this that requires the solution of the equations in a particular order and prevents solution by one vertical strip at a time, as will be described later. Therefore, an alternate method will be used. (This method is being used by the Canadians and British in their semi-implicit models.) Equation (13) can be written

$$C\,\Phi_{yy} + E\,\Phi = F \tag{25}$$

where

$$\Phi = \begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \\ \overline{\phi}_3 \\ \cdot \\ \cdot \\ \cdot \\ \overline{\phi}_L \end{pmatrix} \qquad F = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \cdot \\ \cdot \\ \cdot \\ X_L \end{pmatrix} \tag{26}$$

$$C = \begin{pmatrix} (-a_1 + b_1\Delta p_1) & (-a_2 + 2b_1\Delta p_2) & (-a_3 + 2b_1\Delta p_3) & \ldots(-a_L + 2b_1\Delta p_L) \\ a_1 & (a_2 + b_2\Delta p_2) & (a_3 + 2b_2\Delta p_3) & \ldots(a_L + 2b_2\Delta p_L) \\ -a_1 & -a_2 & (-a_3 + b_3\Delta p_3) & \ldots(-a_L + 2b_3\Delta p_L) \\ a_1 & a_2 & a_3 & \ldots(a_L + 2b_4\Delta p_L) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (-1)^L a_1 & (-1)^L a_2 & (-1)^L a_3 & \ldots[(-1)^L a_L + b_L\, p_L] \end{pmatrix} \tag{27}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 0 & \cdots & 0 \\ 2 & -2 & 1 & \cdots & 0 \\ -2 & 2 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -2(-1)^L & 2(-1)^L & -2(-1)^L & \cdots & 1 \end{bmatrix} \qquad (28)$$

The vertical modes of (25) are separated as follows. Multiply through C^{-1}, giving

$$\Phi_{yy} + C^{-1}E\,\Phi = C^{-1}\,F \qquad (29)$$

Find the eigenvalues and eigenvectors of $(C^{-1}E)^T$

$$(C^{-1}E)^T\,G^{(i)} = \lambda_i G^{(i)} \qquad (30)$$

Multiply (29) by $G^{(i)}$.

$$G^{(i)}.\Phi_{yy} + G^{(i)}.(C^{-1}E\Phi) = G^{(i)}.(C^{-1}F) \qquad (30a)$$

or

$$G^{(i)}.\Phi_{yy} + (C^{-1}E)^T(G^{(i)}.\Phi) = G^{(i)}.(C^{-1}F) \qquad (30b)$$

Because of (30), this can be written as

$$G^{(i)}.\Phi_{yy} + \lambda_i G^{(i)}\Phi = G^{(i)}.(C^{-1}F) \qquad (30c)$$

Or, defining

$$B_i = G^{(i)}.\Phi \qquad (31)$$

and

$$H_i = G^{(i)}.(C^{-1}F)\ , \qquad (31a)$$

equation (30b) can be written

$$B_{iyy} + \lambda_i B_i = H_i,\ \ i = 1,2,\ldots,L \qquad (32)$$

The λ_i are the eigenvalues of $(C^{-1}E)^T$ (or $(C^{-1}E)$ since it is symmetric) and $G^{(i)}$ are the corresponding eigenvectors. The L equations, (29), can be solved independently once the eigenvalues are known. Since C and E only depend upon basic-state parameters, the λ's need be solved for only once. Once (29) has been solved for the B's, the $\overline{\phi}$'s are obtained from

$$\Phi_k = (G^{(k)})^{-1}.B_k \ , \quad k = 1,2,\ldots,L \tag{33}$$

<u>Horizontal Solution for the B's</u>

Equation (29) is written in expanded index form as

$$B_{j-1i} + (\Delta y^2\lambda_i - 2)B_{ji} + B_{j+1i} = \Delta y^2 H_{ji} , \tag{34}$$

This is to be solved with the boundary condition

$$f\ u_k + \tfrac{1}{2}\,\overline{\phi}_{ky} = 0 \tag{35}$$

or

$$B_{iy} = G^{(i)}.\Phi_y = -\ 2\ f\ G^{(i)}.U_i \ . \tag{36}$$

Taking into account the boundary conditions, (36), equation (34) is written in matrix form as

$$\begin{pmatrix} a_{2i} & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & a_{3i} & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & a_{4i} & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & a_{5i} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 & a_{N-1i} \end{pmatrix} \begin{pmatrix} B_{2i} \\ B_{3i} \\ B_{4i} \\ B_{5i} \\ \vdots \\ B_{N-1i} \end{pmatrix} = \begin{pmatrix} \Delta y^2 H_{2i} + \Delta y (B_{iy})_{3/2} \\ \Delta y^2 H_{3i} \\ \Delta y^2 H_{4i} \\ \Delta y^2 H_{5i} \\ \vdots \\ \Delta y^2 H_{N-1i} - y (B_{iy})_{N-\frac{1}{2}} \end{pmatrix} \tag{37}$$

where

$$\begin{aligned} a_{2i} &= \Delta y^2 \lambda_i - 1 \\ a_{ji} &= \Delta y^2 \lambda_i - 2 \ , \quad j = 3,4,\ldots,N-2 \\ a_{N-1i} &= \Delta y^2 \lambda_i - 1 \end{aligned} \tag{40}$$

N is the number of points in the north-south direction. Equation (39) is solved by standard methods for tri-diagonal matrices (e.g., see Isaacson and Keller, 1966, p. 55). Since it is desired to show how the present system of equations can be solved one vertical column at a time, a detailed exposition of the method will be given.

But first, let's put things into perspective. We wish to solve the system (1)-(6), one vertical column at a time. This is the counterpart in this model of solving the general problem one strip at a time, a strip being one row of data extending vertically through the model. The dimensions of a strip would be NxL, where a row length is N and L is the number of model levels. In order to solve (1)-(6) we have formed the equation (13). Equation (13) in turn has been transformed to (32). In effect, the vertical modes of the problem have been separated, and solution for the amplitude of each mode can be obtained separately.

The question is: How can (34) be solved so that we can proceed, knowing only information for one column at a time? To see the answer to this question, we now give a detailed account of the solution to (34). Essentially, the method is one of Gaussian elimination. It is noted that, for each value of i, (34) represents a set of N equations for the B_{ji}'s. This is also clear from the equation in matrix form, (39). The solution proceeds as follows: 1) eliminate B_{2i} from the third through N-1th equations, 2) eliminate B_{3i} from the fourth through N-1th equations, 3) proceed as in 1) and 2) for the B_{4i} through B_{N-2i}. At this point, the last equation can be solved directly for B_{N-1i}. The next to last equation only involves B_{N-1i} and B_{N-2i}. The third from last equation involves only B_{N-1i}, B_{N-2i}, and B_{N-3i}, etc. Therefore, 4) solve recursively for B_{N-1i} B_{N-2i}, B_{N-3i}, ..., B_{2i}, starting from the last equation and ending with the first.

In particular, the process takes the form of calculating recursively for

$$\begin{aligned} \alpha_{2i} &= a_{2i} & \gamma_{2i} &= 1/\alpha_{2i} \\ \alpha_{ji} &= a_{ji} - \gamma_{j-1_i}, & j &= 3,4,\ldots,N-1 \\ \gamma_{ji} &= 1/\alpha_{ji} \quad , & j &= 3,4,\ldots,N-2 \end{aligned} \tag{41}$$

Then solve recursively for

$$\begin{aligned} g_{2i} &= (\Delta y^2 H_{2i} + \Delta y\ V_{iy})/\alpha_{2i} \\ g_{ji} &= (\Delta y^2 H_{ji} - g_{j-1_i})/\alpha_{ji} \ , \quad j = 3,4,\ldots,N-2 \\ g_{N-1i} &= (\Delta y^2 H_{N-1i} - \Delta y V_{iy} - g_{N-2i})/\alpha_{N-1i} \end{aligned} \tag{42}$$

The back-substitution is obtained by

$$\begin{aligned} B_{N-1i} &= g_{N-1i} \\ B_{ji} &= g_{ji} - \gamma_{ji} B_{j+1i} \end{aligned} \tag{43}$$

Now, suppose that we sweep through the data from south to north, one column (or strip in 3D model) at a time. When we reach the northern boundary, B_{N-1i} is known. B_{Ni} is known from the boundary conditions. We can now use (33) to obtain Φ_k at j=N,N-1. Since U^*_k, V^*_k, and G^*_i, are also known for j=N,N-1, we may use (17)-(23) to obtain one column (or strip) of $\overline{(\ )}^{2t}$ variables. Equation (24) is used to obtain a column (or strip) of $(\ )^{\tau+1}$ variables. We may now use (43) to obtain another column (or strip) of $(\ )^{\tau+1}$ variables. When a sufficient number of columns of $(\ )^{\tau+1}$ variables are available, the U^*_k, V^*_k, and G^*_i can be obtained for τ+1. From these, X_k can be obtained for (13), and therefore the right-hand side of (32) can be determined. This means that the forward elimination of (41) and (42) can begin for τ+1 while the back-substitution of (43) for τ proceeds only a few columns (or strips) ahead. Therefore, we have an algorithm for solving the system of equations that requires only a few columns (or strips) of data in the computer storage at one time. No field needs more than four columns (or strips) of data at once. Inputting or outputting of data will occur for only one of these columns (or strips) at a time. This possibility was the big promise that was to be investigated with this model.

It is noted that the α's and γ's of (41) do not vary in time. This means that by saving the γ's (noting that $\alpha_{ji} = 1/\gamma_{ji}$), we could reduce the computation. In the present model, the storage necessary would be small, and likewise the amount of computation saved would be small.

But now let's consider a full 3D model. In such a model, (32) will have the form

$$m^2(B_{kxx} + B_{kyy}) + \lambda_k B_k = H_k, \quad k = 1,2,\ldots,L \qquad (44)$$

where m is the map factor. There are two usual possibilities. Either the model equations are formulated on a polar stereographic grid, or they are formulated on a grid whose map factor changes only in the north-south direction. The relevance of the difference will be seen shortly.

In expanded index notation, (44) can be written

$$B_{k\ ij-1} + [B_{k\ i-1j} + (\lambda^k_{ij} - 4)B_{k\ ij} + B_{k\ i+1j}] + B_{k\ ij+1} = f^k_{ij} \qquad (45)$$

where

$$\lambda^k_{ij} = \frac{\lambda_k}{m^2_{ij}} \Delta^2 \qquad (46)$$

$$f^k_{ij} = \frac{H_k}{m^2_{ij}} \Delta^2 \qquad (47)$$

and Δ is the grid distance. In matrix notation, we can write

$$D^k B^k = F^k \qquad (48)$$

where

$$D^k \quad \begin{pmatrix} E^k & -I & 0 & 0 \ldots \\ -I & E^k & -I & 0 \ldots \\ 0 & -I & E^k & -I \ldots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad E^k_j = \begin{pmatrix} 4-\lambda^k_{ij} & -1 & 0 & \ldots \\ -1 & 4-\lambda^k_{2j} & -1 & \ldots \\ 0 & -1 & 4-\lambda^k_{3j} & \ldots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{pmatrix} \qquad (49)$$

$$B^k = \begin{bmatrix} B^k_{i1} \\ B^k_{i2} \\ B^k_{i3} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad F^k = \begin{bmatrix} -f^k_{i1} \\ -f^k_{i2} \\ -f^k_{i3} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix} \qquad \begin{matrix}(49)\\ \text{cont.}\end{matrix}$$

Formally, (48), (49) look like (39), (40). The difference is that the matrix elements are now matrices themselves. The solution, also, is formally like (41)-(43), except that multiplications are now matrix multiplications, and inverses are matrix inverses. In particular, instead of (41), we have

$$\begin{array}{lll} A^k_1 = E^k_1 & \Gamma^k_1 = -(A^k_1)^{-1} & \\ A^k_j = E^k_j + \Gamma^k_{j+1} \quad , & j = 3,4,\ldots,N-1 & \qquad (50) \\ \Gamma^k_j = -(A^k_j)^{-1} \quad , & j = 3,4,\ldots,N-2 & \end{array}$$

In order to generate the final solution, the Γ's are required. They do not vary in time, so they could be calculated once and saved. In fact, they are generally fairly sparse matrices, with large values along the diagonal and rapidly decreasing values away from the diagonal. If we were to save all the Γ's, we would be saving LxNx(NxN) numbers for an NxN grid. For L=6 and N=65, this is 1,647,750 numbers! For each strip we would need 25,350 numbers in core storage. This would hardly help the I/O problem that this program logic was designed to alleviate! However, we may take advantage of the sparseness of the Γ matrices. It is found that the rate at which the numbers off the diagonal fall to zero is determined largely by the size of the λ_{ij}^{k}'s relative to 1. The bigger the absolute value of the λ_k's (they are negative), the faster the off-diagonal elements fall to zero. The larger λ_k's are related to the slower internal modes on the model, so that the smaller λ_k's of the more rapid gravity wave modes cause the largest off-diagonal elements in the Γ's.

Now, we consider the effect of the map factor on the solution. If m_{ij} varies with both i and j, then each diagonal element of E_j^k differs from the next. The same will be true of the Γ_j^k's and a rather large storage of numbers is needed. For a 6-layer model with an isothermal stratification (determining the λ^k's) and a two times variation in m, it is estimated that the storage requirement is Nx(56N-327). For N=65 this is 215,345 storage locations (words).

If m_{ij} varies only with j, then each diagonal element of E_j^k will be the same, and some storage savings can be obtained. In this case, the estimated storage is $20N^2+95N-1390$ ($N\geq20$). For N=65, this gives 89,285 storage locations.

Now suppose that the grid is rotated so the m_{ij} varies only with i. Then each diagonal element of E_j^k differs from the next, but it is found that the sequence of Γ_j^k's converges to a limit, quite rapidly for the slow modes, and less rapidly for the faster modes. The savings in required storage for this case is considerable. It is estimated that Nx365 ($N\geq20$) storage locations are needed, or 7300 for N=65. For completeness, it is mentioned that for a constant map factor, we could get away with saving 5071 numbers.

In summary, it appears that the storage requirements are extreme for general map factor variations, but not too severe for models on a Mercator, spherical, or similar grids. This presupposes that a method could be found to store only the required elements of the Γ_j^k's, to catalog them, and to make the required multiplications of the Γ_j^k's with other matrices. On a vector machine, it is suspected that the procedure would be extremely inefficient.

A Note on Accuracy

In the discussion of the last section, it has been assumed that the elements of the Γ matrices are neglected if they are 10^{-4} the magnitude of the corresponding diagonal elements. There are N multiplications and N-1 additions required to get each B_{ij} from B_{j+1i}. The maximum relative error during this process should be $2(N-1)x10^{-4}$. (See Henrici, 1964, p. 169.) B_{N-2i} can be expected to have this error, while B_{N-3i} would have twice this error, etc. The average error would be $\frac{1}{2}(N-1)2(N-1)x10^{-4}$ or $(N-1)^2x10^{-4}$. For N=65, the average relative error (maximum) is 0.41! Surely the storage estimates have been on the low side.

There is another place where accuracy is a consideration. In (4) the factor β has been included. For the Helmholtz equation (13), this determines the presence or absence of G_k in X_k. Since this is the dominant term of X_k, the difference is important. If G_k is present, we solve for the total ϕ-field and great accuracy is needed in the solution of (13) or (32). Double precision was found to be necessary on the IBM 360/195 with N=10! If, however, $\beta=0$ so that G_k is not included in X_k, then single precision is sufficient, even for N=100.

Another Possibility with Variable Map Factor

We consider next an iterative solution of (44) suggested by J. Gerrity (personal communication). Expand the map factor as a constant plus a deviation:

$$m^2 = \bar{m} + \Delta \tag{51}$$

Then (44) can be written

$$B_{kxx} + B_{kyy} + \frac{\lambda_k}{\bar{m}} B_k = \frac{1}{\bar{m}}[H_k - \Delta(B_{kxx} + B_{kyy})] \tag{51a}$$

If we calculate the right-hand side values from the previous iteration, this can be written

$$\begin{aligned}(B_{kxx} + B_{kyy})^{n+1} + \frac{\lambda_k}{\bar{m}} B_k^{n+1} &= \frac{1}{\bar{m}} [H_k - \Delta(B_{kxx} + B_{kyy})^n] \\ &= \frac{1}{\bar{m}^2} [H_k + \frac{\Delta}{\bar{m}} \lambda_k B_k^n]\end{aligned} \tag{52}$$

where n is the iterative count. The convergence of this iterative procedure can be investigated by considering a particular wave component, for which

$$(\ )_{xx} + (\ )_{yy} = - K^2(\ ) \tag{53}$$

Letting the first guess for the solution be B_k^o, the solution to (52) is found to be

$$B^n = (\beta \frac{\Delta}{\bar{m}} \lambda)^n B^o + \beta H_k \sum_{\ell=0}^{n-1} (\beta \frac{\Delta}{\bar{m}} \lambda)^\ell \tag{54}$$

where

$$\beta = [m^2\left(- K^2 + \frac{\lambda}{\bar{m}}\right)]^{-1} \qquad (55)$$

Convergence of the scheme is guaranteed by $\left|\beta \frac{\Delta}{\bar{m}}\right| < 1$, or

$$\left|\frac{\Delta}{\bar{m} + \Delta}\right| < \left(- \frac{K^2\ \bar{m}}{\lambda} + 1\right) \quad (\lambda < 0) \qquad (56)$$

Clearly, this is always satisfied if $\bar{m}$ is chosen within the range of m^2, i.e., if $|\Delta| < m^2$.

It is noted from (54) that as $n\to\infty$, the first term vanishes. Therefore, the final solution is unaffected by the first guess. A good first guess can speed the convergence, however. This cas be seen easily by letting $B_k^o \doteq \beta\ H_k \sum\limits_{\ell=0}^{\infty} \left(\beta \frac{\Delta}{\bar{m}} \lambda\right)^{\ell}$ in (54).

If (52) is used to solve (44), then the horizontal solution procedure shown earlier cannot be followed. Instead, we must complete the solution for B_k for the entire grid before we can march ahead in time. The solution can still be performed in strips, however. Further, the coefficients on the left-hand side of (52) are constant and uniform in space so that the storage necessary for the matrix inverses used in the solution to (52) is minimal. Using B_k from the previous time step for the first guess should provide a first guess accurate enough to give convergent solutions in only a few scans. It therefore appears that an efficient method exists for the solution of (44) with variable map factor. And the method allows all I/O in strips with potential increase in I/O efficiency.

Summary

The equations for a semi-implicit model in pressure coordinates have been outlined. The method of solution for constant map factor has been given in some detail. And two possible methods of solution have been given for variable map factor. In all cases, the core storage is assumed to be arranged in strips. At no time during the solution is it necessary to use 2D horizontal fields. The feasibility of totally stripped program logic is considered to be shown. However, actual calculations have not been made which prove this feasibility.

The 2D model written for this study and its explicit version can be used in studies of geostrophic adjustment, and other problems.

References

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